

AVERAGES OVER CURVES WITH TORSION

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ABSTRACT. We establish L^p Sobolev mapping properties for averages over certain curves in \mathbb{R}^3 , which improve upon the estimates obtained by $L^2 - L^\infty$ interpolation.

Let T be the operator given by convolution in \mathbb{R}^3 against a smooth cutoff of arclength measure on the helix $\gamma(t) = (\cos t, \sin t, t)$,

$$Tf(x) = \int f(x_1 - \cos t, x_2 - \sin t, x_3 - t) \phi(t) dt.$$

For $1 < p < \infty$, let $H^{s,p}(\mathbb{R}^3)$ denote the nonhomogeneous Sobolev space consisting of functions in $L^p(\mathbb{R}^3)$ whose fractional derivative of order s also lies in $L^p(\mathbb{R}^3)$. We consider the following question:

For which values of s (depending on p) does $T : L^p(\mathbb{R}^3) \rightarrow H^{s,p}(\mathbb{R}^3)$?

By duality, it suffices to consider $2 \leq p < \infty$. As shown by the first two authors in [OS], a necessary condition is that

$$s \leq \frac{1}{6} + \frac{1}{3p} \quad \text{if } 2 \leq p \leq 4,$$

$$s \leq \frac{1}{p} \quad \text{if } 4 \leq p < \infty.$$

Simple arguments (see for example the lemma below) show that $T : L^2(\mathbb{R}^3) \rightarrow H^{\frac{1}{3},2}(\mathbb{R}^3)$. Interpolation with the trivial $L^\infty(\mathbb{R}^3)$ boundedness of T yields a sufficient condition of $s \leq \frac{2}{3p}$. In particular, interpolation yields

$$(1) \quad T : L^4(\mathbb{R}^3) \rightarrow H^{\frac{1}{6},4}(\mathbb{R}^3).$$

In this note, we combine the arguments of [OS] with Bourgain's [B] improvement of the conic square function estimate of Mockenhaupt [M] to obtain the following.

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Theorem. *There exists $\sigma > 0$ such that*

$$(2) \quad T : L^4(\mathbb{R}^3) \rightarrow H^{\frac{1}{6}+\sigma,4}(\mathbb{R}^3).$$

We should point out that T is a model for curve-averaging operators whose canonical relations have two-sided Whitney folds. In two dimensions these operators are much easier to analyze and optimal results are known. See e.g., [SS] and [SW].

In three dimensions, the translation invariant operators of this type are the averages over curves with non-vanishing torsion (a curve $\gamma(t)$ has non-vanishing torsion if the vectors $\{\gamma'(t), \gamma''(t), \gamma'''(t)\}$ are linearly independent for each t .) The helix and the twisted cubic, $\gamma(t) = (t, t^2, t^3)$, are basic examples. We restrict attention here to the helix since this operator has the light cone in ξ as its folding set. A modification of Bourgain's estimate to conic hypersurfaces with one non-vanishing principle curvature would yield the theorem for general curves with torsion.

The value of σ is related to the exponent τ in equation (132) of [B], which is not explicitly determined. Any $\sigma < \frac{1}{3}\tau$ works. In particular, an optimal value $\tau = \frac{1}{4}$ would yield the nearly optimal condition $\sigma < \frac{1}{12}$. Recently, Tao and Vargas [TV] have modified Bourgain's arguments and obtained a definite value of τ . The authors would like to thank T. Tao for a helpful conversation regarding Bourgain's work.

To begin the proof of (2), let

$$(3) \quad \widehat{T}(\xi) = \int e^{-i\xi_1 \cos t - i\xi_2 \sin t - i\xi_3 t} \phi(t) dt$$

denote the Fourier multiplier associated to T .

Let $\xi' = (\xi_1, \xi_2)$. The oscillatory integral (3) has no critical points for $|\xi'| < |\xi_3|$. The following thus holds.

$$|\widehat{T}(\xi)| = \mathcal{O}(|\xi|^{-N}) \quad \forall N, \quad \text{if } |\xi'| \leq .99 |\xi_3|.$$

For $|\xi'| > |\xi_3|$ there are two, nondegenerate critical points. The following is thus a consequence of Van der Corput's Lemma,

$$|\widehat{T}(\xi)| \leq C |\xi|^{-\frac{1}{2}}, \quad \text{if } |\xi'| \geq 1.01 |\xi_3|.$$

A simple interpolation argument implies (2) for the operator obtained by conically restricting $\widehat{T}(\xi)$ to either of the above regions. Indeed, since these bounds imply that these two localized pieces gain a $1/2$ -derivative on L^2 , the interpolation argument behind (1) yields estimates of the form (2) for each term with the desired $\sigma = 1/12$.

It thus suffices to establish (2) for the operator S obtained by restricting the multiplier $\widehat{T}(\xi)$ to the region A , defined by $.98 \leq |\xi'|/|\xi_3| \leq 1.02$, via a smooth conic cutoff. Let S_λ denote the operator obtained by further restricting to the region $\lambda \leq |\xi_3| \leq 2\lambda$. The theorem is then a result of showing that, for some number $a > 0$, for all $\lambda > 2$,

$$(4) \quad \|S_\lambda\|_{4,4} \leq C (\log \lambda)^a \lambda^{-\frac{1}{6}-\frac{\sigma}{3}}.$$

We restrict attention to $\xi_3 > 0$. Following [OS], we make a further decomposition of S_λ by decomposing the conic set A into a union of conic sets A_λ^j as follows:

$$\text{for } j \geq 1, \text{ set } A_\lambda^j = \{1 + 2^{j-1} \lambda^{-\frac{2}{3}} \leq |\xi'|/|\xi_3| \leq 1 + 2^j \lambda^{-\frac{2}{3}}\};$$

$$\text{set } A_\lambda^0 = \{1 - \lambda^{-\frac{2}{3}} \leq |\xi'|/|\xi_3| \leq 1 + \lambda^{-\frac{2}{3}}\};$$

$$\text{for } j \leq -1, \text{ set } A_\lambda^j = \{1 - 2^{|j|} \lambda^{-\frac{2}{3}} \leq |\xi'|/|\xi_3| \leq 1 - 2^{|j|-1} \lambda^{-\frac{2}{3}}\}.$$

Introducing a suitable partition of unity on the Fourier transform side leads to the decomposition

$$S_\lambda = \sum_j S_\lambda^j.$$

Inequality (4) will follow from

$$(5) \quad \|S_\lambda^j\|_{4,4} \leq C (\log \lambda)^a \lambda^{-\frac{1}{6} - \frac{\tau}{3}} 2^{\frac{|j|}{2}(\tau - \frac{1}{4})}$$

for all j and λ . At this point we make a further decomposition as in [M] of A_λ^j into sets A_λ^{jm} supported in ξ' sectors of angle $\delta \doteq 2^{|j|/2} \lambda^{-\frac{1}{3}}$. This leads to a decomposition

$$S_\lambda^j = \sum_{m=1}^{\delta^{-1}} S_\lambda^{jm}.$$

In the notation of Theorem 1.0 of [M], we have

$$\widehat{S}_\lambda^{jm}(\xi) = \widehat{\psi}_m(\lambda^{-1}\xi', \lambda^{-1}(1 + \delta^2)\xi_3) \widehat{T}(\xi).$$

The quantity N of that theorem is related to j and λ by $N = \delta^{-1}$.

Lemma.

$$\|S_\lambda^{jm}\|_{4,4} \leq C \lambda^{-\frac{1}{4}} \delta^{\frac{1}{4}}.$$

Proof. The proof is almost identical to that of the Lemma in [OS], and is obtained by interpolating the following estimates

$$(6) \quad \begin{aligned} \|S_\lambda^{jm}\|_{2,2} &\leq C (\lambda \delta)^{-\frac{1}{2}}, \\ \|S_\lambda^{jm}\|_{\infty,\infty} &\leq C \delta. \end{aligned}$$

The first estimate in (6) is the bound $|\widehat{S}_\lambda^j(\xi)| \leq C (\lambda \delta)^{-\frac{1}{2}}$, which follows from Van der Corput's Lemma as shown in [OS]. For the second estimate, we consider the term m corresponding to the ξ' sector along the negative ξ_2 axis. The convolution kernel of S_λ^{jm} , written in the new coordinates

$$(y_1, y_2, y_3) = (x_1, x_2 + \alpha x_3, \alpha x_3 - x_2), \quad \alpha = (1 + \delta^2)^{-1},$$

takes the form

$$K(y) = \lambda^3 \delta^3 \int \phi(t) \theta(\lambda \delta (y_1 - \cos t), \lambda \delta^2 (y_2 - \sin t - \alpha t), \lambda (y_3 + \sin t - \alpha t)) dt.$$

Here and below, θ denotes a Schwartz function with seminorms bounded independent of j, m, λ , and with $\widehat{\theta}(\eta) = 0$ for $|\eta_3| \leq 1$. We need to show that $\|K\|_{L^1} \leq C \delta$, and may thus replace $\phi(t)$ by $\phi_\delta(t)$ which vanishes for $|t| \leq 10\delta$. We write $\theta = \partial_3 \theta$ for some new θ to express $K(y)$ as

$$\lambda^2 \delta^3 \int \left(\frac{\phi_\delta(t)}{\alpha - \cos t} \right)' \theta(\dots) dt + \lambda^3 \delta^4 \int \frac{\sin t \phi_\delta(t) \theta(\dots)}{\alpha - \cos t} dt + \lambda^3 \delta^5 \int \frac{(\alpha + \cos t) \phi_\delta(t) \theta(\dots)}{\alpha - \cos t} dt.$$

The inequality $\alpha - \cos t \geq t^2/10$ for $|t| \in [10\delta, \pi]$, together with $|\phi'_\delta(t)| \leq C\delta^{-1} \leq C\lambda^{1/3}$, yields the desired $L^1(dy)$ norm bounds on the first and third terms. The desired bound for the second term follows by a further integration by parts of the same kind. \square

To conclude the proof of (5), we apply Bourgain's estimate (132) of [B] to obtain

$$\left\| \sum_m S_\lambda^{jm} f \right\|_4 \leq C \delta^{\tau - \frac{1}{4}} \left\| \left(\sum_m |S_\lambda^{jm} f|^2 \right)^{\frac{1}{2}} \right\|_4.$$

The number of indices m is $O(\delta^{-1})$, so

$$\sum_m |S_\lambda^{jm} f(x)|^2 \leq C \delta^{-\frac{1}{2}} \left(\sum_m |S_\lambda^{jm} f(x)|^4 \right)^{\frac{1}{2}}.$$

With \widehat{f}_m representing the localisation of \widehat{f} to an appropriate sector in ξ' , we thus have

$$\begin{aligned} \left\| \sum_m S_\lambda^{jm} f \right\|_4 &\leq C \delta^{\tau - \frac{1}{2}} \left\| \left(\sum_m |S_\lambda^{jm} f|^4 \right)^{\frac{1}{4}} \right\|_4 \\ &\leq C \lambda^{-\frac{1}{6} - \frac{\tau}{3}} 2^{\frac{|j|}{2}(\tau - \frac{1}{4})} \left\| \left(\sum_m |f_m|^4 \right)^{\frac{1}{4}} \right\|_4 \\ &\leq C \lambda^{-\frac{1}{6} - \frac{\tau}{3}} 2^{\frac{|j|}{2}(\tau - \frac{1}{4})} \left\| \left(\sum_m |f_m|^2 \right)^{\frac{1}{2}} \right\|_4. \end{aligned}$$

A result of Córdoba [C] gives

$$\left\| \left(\sum_m |f_m|^2 \right)^{\frac{1}{2}} \right\|_4 \leq C |\log \delta|^a \|f\|_4$$

for some positive a , which completes the proof of (5).

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